

Dyonic Reissner-Nordström black hole: extended Dirac quantization from 5D invariants.

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Abstract

The aim of present work is to extend the application of Weitzböck Induced Matter Theory (WIMT) to a dyonic Reissner-Nordström Black Hole (RNBH), by proposing a condition compatible with a quantization relation between gravitational mass and both magnetic and electric charges from a geometric product defined as an invariant in 5D.

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I. INTRODUCTION

In a previous work [1] we have developed an extension of the Induced Matter Theory (IMT) [2] in presence of a Weitzenböck geometry, called Weitzenböck Induced Matter Theory (WIMT). The spirit of IMT is that ordinary matter and physical fields present in our $4D$ universe can be geometrically induced from a $5D$ space-time which is at least Ricci-flat (in the sense of Riemannian connections) with a non-compact extra dimension, so that the $5D$ physical vacuum defined by the Ricci-flatness condition supports the Campbell-Magaard embedding theorem as a particular case of ${}^{(5D)}R_{ab} = \lambda {}^{(5D)}g_{ab}$, in which $\lambda = 0$. This makes possible to define a $5D$ apparent vacuum that determines the equations of motion for the fields of the theory. The aim of WIMT is to use an alternative description, based in the Weitzenböck geometry, to apply an IMT-like formalism for any $5D$ space-time even if this is not flat in the Riemannian sense. With the WIMT one can make the geometrical description of a problem from a non-flat (but torsion-less) Riemannian manifold using the Levi-Civita connections, into a Weitzenböck's geometrical description with a Riemann-Weitzenböck null tensor, because the Riemann-Weitzenböck tensor is the Riemann curvature tensor expressed in terms of Weitzenböck connections: ${}^{(W)}\Gamma_{dc}^a$. In a coordinate (holonomic) basis the Riemann-Weitzenböck curvature takes the form

$$\begin{aligned} {}^{(W)}R_{bcd}^a &= {}^{(W)}\Gamma_{dc,b}^a - {}^{(W)}\Gamma_{db,c}^a + {}^{(W)}\Gamma_{dc}^n {}^{(W)}\Gamma_{nb}^a \\ &\quad - {}^{(W)}\Gamma_{db}^n {}^{(W)}\Gamma_{nc}^a. \end{aligned} \quad (1)$$

In this case we can use the IMT tools to induce $4D$ effective space-time dynamics by making a foliation over the Riemann-Weitzenböck flat $5D$ space-time on which we define a $5D$ vacuum in the Weitzenböck sense

$${}^{(W)}R_{bcd}^a = 0. \quad (2)$$

Once this choice has been made, we can recover the Riemannian description by doing a transformation over the induced tensors (now obtained in terms of Weitzenböck connections) according to ${}^{(LC)}\Gamma_{bc}^a = {}^{(W)}\Gamma_{bc}^a + K_{bc}^a$. Here ${}^{(LC)}\Gamma_{bc}^a$ is the symbol denoting the LC connection (the usual second kind Christoffel symbols) and K_{bc}^a is the contortion, which depends on the non-metricity and torsion.

In the context of WIMT we have studied gravito-electromagnetic dynamics and magnetic monopoles on both, astrophysical and cosmological scenarios[3]. The Weitzenböck torsion

must be viewed as a key element in order to construct a non-symmetric connection capable to avoid the $d(d(A)) = 0$ dynamic condition, who eliminates the magnetic monopoles from the theory. The cotangent (penta-vector) A is a 5D 1-form characterized by $\underline{A} = A_n \underline{e}^n$, $n = 0...4$, with components given by $A_0 = \phi$ (the electric scalar potential), $A_i = \mathbf{A}_i$ being the i -th component of the 3D vector potential and $A_4 \equiv \psi$ the gravitational scalar potential. The gravito-magnetic dynamic equations in terms of exterior covariant derivatives ¹, are

$$^{(m)}J = *d(F) = *d(d(A)), \quad (3)$$

with the Faraday 2-form defined by $F = d(A)$. The asterisk denotes the duality operator, which, in 5D, originates the apparition of the 3-form $*F$ which is the dual of F in the expression (3). This is very clear in next equations, especially in Eq. (6). On the other hand results obvious that $d(d(A)) = 0$ implies $*d(d(A)) = 0$. Indeed, for a general affine

¹ A p -form is a tensor object which we call W in present footnote, this is p times cotangent and totally antisymmetric

$$W = \frac{1}{p!} w_{i_1 \dots i_p} \underline{e}^{i_1} \wedge \dots \wedge \underline{e}^{i_p},$$

in which wedge product is the complete anti-symmetrization of the tensorial product. The exterior covariant derivative is linked to a covariant derivative by

$$d(W) = \frac{1}{p!} w_{i_1 \dots i_p; k} \underline{e}^k \wedge \underline{e}^{i_1} \wedge \dots \wedge \underline{e}^{i_p},$$

each different covariant derivative ; defines a different exterior derivative. The adjoint operation denoted by $*$ is defined in a manifold of dimension m with the expression

$$*W = \frac{\sqrt{|g|}}{(m-p)!p!} \varepsilon_{j_1 \dots j_p i_{p+1} \dots i_m} w^{j_1 \dots j_p} \underbrace{\underline{e}^{i_{p+1}} \wedge \dots \wedge \underline{e}^{i_m}}_{m-p},$$

then adjoint operation interchanges tensor order from p to $m-p$. Taking last three equations we must write the magnetic current of the Maxwell equations as in 3. For more details, the reader can see [4].

connection, we obtain

$$d(d(A)) := d(A_{n;m} \underline{e}^m \wedge \underline{e}^n) = \quad (4)$$

$$\begin{aligned} &= A_{n;m;p} \underline{e}^p \wedge \underline{e}^m \wedge \underline{e}^n = \\ &= \{ \vec{e}_p(\vec{e}_n(A_m)) - \vec{e}_p(A_l \Gamma_{nm}^l) \\ &\quad - (\vec{e}_k(A_m) - A_l \Gamma_{km}^l) \Gamma_{np}^k \\ &\quad - (\vec{e}_n(A_k) - A_l \Gamma_{nk}^l) \Gamma_{mp}^k \} \underline{e}^p \wedge \underline{e}^m \wedge \underline{e}^n, \end{aligned} \quad (5)$$

$$\begin{aligned} *d(d(A)) &:= \left([*F]_{abp} \right)^{;p} e^a \wedge e^b \\ &= \left(\frac{\sqrt{|g|}}{2!3!} \epsilon_{abnmp} g^{ms} g^{pd} A_{;sd}^n \right) e^a \wedge e^b, \end{aligned} \quad (6)$$

where

$$\begin{aligned} A_{;sd}^n &= e_d(e_s(A^n)) + e_d(A^l \Gamma_{ls}^n) \\ &\quad + (e_s(A^m) + A^l \Gamma_{ls}^m) \Gamma_{md}^n \\ &\quad - (e_m(A^n) + A^l \Gamma_{lm}^n) \Gamma_{sd}^m. \end{aligned} \quad (7)$$

It is easy to see that for a coordinate basis $\{\vec{e}_n\}$, of the $5D$ tangent space $5DTM$, the first term of Eq. (4) vanishes, and the last three terms are proportional to the product of the shape $\Gamma_{nm}^l \underline{e}^n \wedge \underline{e}^m$, in which the wedge product is (n, m) -antisymmetric. The entire term results to be zero if the connection Γ_{nm}^l is (n, m) -symmetric (as is the case for the Christoffel symbols). Then Eq. (4) and Eq. (6) are both zero, and in such case there are no magnetic monopoles. In order to relax the last condition has been proposed [5] that F to be not an exact 2-form in the entire space-time, in order to obtain $F \neq d(A)$ in some points where are located the magnetic monopoles. In our case, for an $5D$ orthogonal coordinate basis, we keep the relationship $^{(LC)}\nabla_b F^{nb} = J^n$, and the dual equations are $^{(LC)}\nabla_b (*F)^{nmb} = 0$. However, when we rewrite these equations in terms of a Weitzenböck geometric representation, results so that these equations expressed in terms of divergence applied to the Weitzenböck Maxwell tensor and its dual tensor inhomogeneous:

$$^{(W)}\nabla_b^{(W)} F^{nb} = ^{(ge)}J^n, \quad (8)$$

$$^{(W)}\nabla_c^{(W)} \mathcal{F}^{abc} = ^{(gm)}J^{ab}, \quad (9)$$

such that the gravito-electric and gravito-magnetic currents $(^{(ge)}J^n, ^{(gm)}J^{ab})$, obtained from the Weitzenböck representation of a 5D gravito-electrodynamical theory, are given by

$$\begin{aligned} (^{ge})J^n &= - \left[({}^{(W)}F^{mb} + g^{rm} A^p K_{pr}^b - g^{rb} A^p K_{pr}^m) K_{mb}^n \right. \\ &\quad \left. + ({}^{(W)}F^{nm} + g^{rn} A^p K_{pr}^m - g^{rm} A^p K_{pr}^n) K_{mb}^b \right], \end{aligned} \quad (10)$$

$$\begin{aligned} (^{gm})J^{ab} &= \left[\frac{\sqrt{|g|}}{2!} \epsilon_{abcde} g^{ck} g^{dl} g^{es} \{ (A_{n,l} - A_{l,n}) K_{sk}^n \right. \\ &\quad - (A_{s,n} - A_{n,s}) K_{lk}^n + A_m (K_{[sl]}^m + K_{[nl]}^m) {}^{(W)}\Gamma_{sk}^n \\ &\quad \left. + K_{[sn]}^m {}^{(W)}\Gamma_{lk}^n \} \right], \end{aligned} \quad (11)$$

such that we can define the components of gravito-magnetic currents as $(^{gm})J^a = (^{gm})J^{ab} \bar{U}_b$, such that \bar{U}_b are the covariant components of the relativistic penta-velocity that comply with: $\frac{d\bar{U}^a}{ds} + ({}^{(W)}\Gamma_{dc}^a \bar{U}^d \bar{U}^c = 0$. Therefore, the Maxwell tensor in the Weitzenböck representation is: $({}^{(W)}F_{ab} = ({}^{(W)}\nabla_a A_b - ({}^{(W)}\nabla_b A_a$, such that $({}^{(W)}\nabla_a A_b = A_{b,a} - ({}^{(W)}\Gamma_{ab}^c A_c$, with $({}^{(W)}\Gamma_{ab}^c = ({}^{(LC)}\Gamma_{ab}^c - K_{ab}^c$, and the dual Maxwell tensor in the Weitzenböck representation is $({}^{(W)}\mathcal{F}^{nmb} = \frac{\sqrt{|g|}}{2!} \epsilon^{nmbac} ({}^{(W)}F_{ac}$. The contortion K_{ab}^c is originated in the fact that the Weitzenböck connections have an antisymmetric contribution.

The Big Bang cosmology predicts that at very large number of heavy, stable magnetic monopoles should have been produced in the very early universe. However, magnetic monopoles have never been observed, so if they exist at all, they are much rarer than the Big Bang theory predicts, or they should be in causally disconnected regions of the universe, as for example: inside of a black-hole. In this work we shall connect the Christoffel with the Weitzenböck geometrical representations as was realised in a previous work[6], but extending this approach with the aim to introduce 4D magnetic monopoles in the Weitzenböck representation from a 5D Weitzenböck vacuum for a Dyonic Reissner Nordström Black Hole (DRNBH). DRNBH differentiates from standard RNBH in that the first ones have both kind of charges, electric and magnetic, inside the horizon. As was demonstrated in [6], the

imposed conditions over the 5D relativistic invariants there studied², prohibit the existence of a DRNBH, unless that densities of gravito-magnetic mass and the gravito-electric charges to be equals: $\rho_M = \rho_e$, so that they only admit naked singularities. For this reason, we shall try to solve this problem by treating the invariants proposed in [6] as a particular set of values, which must be obtained from an invariant arising from the geometric product of the currents.

II. DYONIC REISSNER-NORDSTRÖM BLACK HOLE (DRNBH) AND EXTENDED DIRAC QUANTIZATION FROM GEOMETRIC PRODUCT.

To address the problem of a DRNBH our propose is to start from an extended 5D DRNBH metric, given by

$$\begin{aligned} {}^{(5D)}ds^2 &= f(r) dt^2 - [f(r)]^{-1} dr^2 \\ &\quad - r^2 (d\theta^2 - \sin^2 \theta d\varphi^2) - dl^2, \end{aligned} \quad (16)$$

where $f(r) = \left[1 - \frac{2M}{r} + \frac{(Q_e^2 + Q_m^2)}{r^2}\right]$. This metric does not provide us a 5D Riemann vacuum, but is a Weitzenböck one. The effective 4D DRNBH[7] is obtained by making a constant foliation over the extra dimension: $l = l_0$

$$\begin{aligned} {}^{(4D)}ds^2 &= f(r) dt^2 - [f(r)]^{-1} dr^2 \\ &\quad - r^2 (d\theta^2 - \sin^2 \theta d\varphi^2), \end{aligned} \quad (17)$$

such that the radiuses are

$$r_{\pm} = M \pm \sqrt{M^2 - [Q_e^2 + Q_m^2]}, \quad (18)$$

² The 5D invariants proposed in [6], are

$$\begin{aligned} {}^{(m)}J \stackrel{\rightrightarrows}{=} \left({}^{(e)}\vec{J}, \vec{U} \right) &:= {}^{(e)}J^a {}^{(m)}J_{ab} U^b = {}^{(m)}J_a {}^{(e)}J^a, \end{aligned} \quad (12)$$

$${}^{(m)}J \stackrel{\rightrightarrows}{=} \left(\vec{U} \right) = {}^{(m)}J_a U^a, \quad (13)$$

$${}^{(e)}J \stackrel{\rightrightarrows}{=} \left(\vec{U} \right) = {}^{(e)}J_a U^a, \quad (14)$$

$$\begin{aligned} {}^{(gem)}J^2 &:= \left({}^{(e)}J \wedge {}^{(m)}J \right) \left({}^{(e)}\vec{J} \wedge {}^{(m)}\vec{J} \right) \\ &= {}^{(gem)}J_{ab} {}^{(gem)}J^{ab}, \end{aligned} \quad (15)$$

where ${}^{(gem)}J \stackrel{\rightrightarrows}{=} {}^{(ge)}J \wedge {}^{(gm)}J$ are the gravito-electro-magnetic 2-form, and U^b are the components of the penta-velocities of the observers.

such that the inner horizon is in r_- . In order to define grovito-electromagnetic invariants with the sources (or currents), we shall think ${}^{(ge)}\underline{J} {}^{(gm)}\underline{J}$ as a geometric product given by both, inner and wedge products of ${}^{(ge)}\underline{J}$ and ${}^{(gm)}\underline{J}$. The first one is the symmetric part of the geometric product and the second one is its antisymmetric part[8]:

$${}^{(ge)}\underline{J} {}^{(gm)}\underline{J} = {}^{(ge)}\underline{J} \cdot {}^{(gm)}\underline{J} + {}^{(ge)}\underline{J} \wedge {}^{(gm)}\underline{J}. \quad (19)$$

The first term in the right side is a scalar and the second one is a 2-form (or bi-covector). In general, we must write a scalar according to ³

$$\begin{aligned} ({}^{(ge)}\underline{J} {}^{(gm)}\underline{J})^2 &= \overrightarrow{g}({}^{(ge)}\underline{J}, {}^{(gm)}\underline{J})^2 + ({}^{(ge)}\underline{J} \wedge {}^{(gm)}\underline{J})^2 \\ &= ({}^{(ge)}J_n {}^{(gm)}J^n)^2 + ({}^{(gem)}J_{nm} {}^{(gem)}J^{nm}), \end{aligned} \quad (22)$$

which is a particular application of Eq. (21). In [6] was proposed the that

$$\underbrace{({}^{(ge)}\underline{J} {}^{(gm)}\underline{J})^2}_{n^2} = \underbrace{({}^{(ge)}J_n {}^{(gm)}J^n)^2}_0 + \underbrace{({}^{(gem)}J_{nm} {}^{(gem)}J^{nm})}_{n^2}, \quad (23)$$

This proposal works good when the gravito-electric and gravito-magnetic charges are in space-time regions causally disconnected, as is the case of a RNBH. However, there are problems in cases when the charges co-exist in causally connected regions of the space-time.

³ For a general multi-tensor object

$$\begin{aligned} T &= A + B_n \underline{e}^n + C_{nm} \underline{e}^n \otimes \underline{e}^m \\ &+ D_{nmp} \underline{e}^n \otimes \underline{e}^m \otimes \underline{e}^p + \dots \end{aligned} \quad (20)$$

where A, B_n, C_{nm}, D_{nmp} are arbitrary scalar functions in $\mathfrak{F}(M)$. We must obtain a scalar defined by

$$\begin{aligned} T^2 &= A^2 + B_n B_{n'} \overrightarrow{g}(\underline{e}^n, \underline{e}^{n'}) \\ &+ C_{nm} C_{n'm'} \overrightarrow{g}(\underline{e}^n, \underline{e}^{n'}) \overrightarrow{g}(\underline{e}^m, \underline{e}^{m'}) \\ &+ D_{nmp} D_{n'm'p'} \overrightarrow{g}(\underline{e}^n, \underline{e}^{n'}) \overrightarrow{g}(\underline{e}^m, \underline{e}^{m'}) \overrightarrow{g}(\underline{e}^p, \underline{e}^{p'}) + \dots \\ &= A^2 + B_n B_{n'} g^{nn'} + C_{nm} C_{n'm'} g^{nn'} g^{mm'} \\ &+ D_{nmp} D_{n'm'p'} g^{nn'} g^{mm'} g^{pp'} + \dots, \end{aligned} \quad (21)$$

which is a generalized expression of a inner product for multi-tensorial objects of any kind. We must notice that such product coincides with the usual inner product for a pure vector or co-vector.

This is the case in a DRNBH when $Q_m \neq 0$. This suggests that, in cases when gravito-magnetic and gravito-electric charges co-exist in the same causally connected space-time, we must require that the co-vector currents must not be orthogonal

$${}^{(ge)}J_n {}^{(gm)}J^n \neq 0. \quad (24)$$

Therefore, in the cases where the source has both, gravito-magnetic and gravito-electric charges, the currents cannot be considered as orthogonal, and

$$({}^{(ge)}\underline{J} {}^{(gm)}\underline{J})^2 = n^2, \quad (25)$$

According to Eq. (24), the first term in r.h.s. of (23) is the square of a simple scalar let's say $({}^{(ge)}J_n {}^{(gm)}J^n)^2 = m^2$, and using this expression with (25), in (23)⁴, we obtain

$$\rho_e \rho_m = \frac{\sqrt{n^2 - m^2}}{2}, \quad (27)$$

in which n^2 is the square of the norm of the geometric product of the currents and m^2 is nonzero due to the non-orthogonality of such currents. If the gravito-electric and gravito-magnetic currents were mutually orthogonal then would be $m^2 = 0$ and (27) would reduce to the already known quantization relation obtained in [6]. In general, if we assume that $U^1, U^5 \neq 0$, we obtain

$$\rho_e = \frac{m^2 U^5}{(U^5)^2 - (U^1)^2}, \quad (28)$$

$$\rho_M = \frac{m^2 U^1}{(U^5)^2 - (U^1)^2}, \quad (29)$$

$$\rho_m = \frac{\sqrt{n^2 - m^2}}{2} \left[\frac{(U^5)^2 - (U^1)^2}{m^2 U^5} \right]. \quad (30)$$

Since $M^2 - Q_e^2 > 0$, one must require $|U^1| > |U^5|$, so that this condition enables us to obtain $\rho_m \neq 0$ in the presence of a true DRNBH⁵. Hence, for a 4D space-time obtained

⁴ The reader can see the Sect. 3 of [6], where we demonstrated that

$$[{}^{(gm)}J_A] = (-\rho_m U^5, 0, 0, 0, \rho_m U^1), \quad \text{with} \quad {}^{(gm)}J_i = 0,$$

and

$$[{}^{(ge)}J_A] = (\rho_e, 0, 0, 0, \rho_M). \quad (26)$$

⁵ The radius remains real for $\frac{4m^8}{n^2 - m^2} > \frac{U^{52} - U^{12}}{U^5}$, which reduces to $\frac{4m^8}{n^2 - m^2} > 1$ over the horizon.

by a constant foliation ⁶, it holds that $g_{\alpha\beta}U^\alpha U^\beta = g_{11}U^1U^1 = 1$. Then, the induced four-velocity is completely given by $U^1 = \sqrt{1 - \frac{2M}{r} + \frac{(Q_e^2 + Q_m^2)}{r^2}}$. On the other hand, from the corresponding 5D geodesic equation for the 5D DRNBH-like metric, we obtain that $U^5 = K(s)$, which is a constant of the affine parameter.

III. FINAL COMMENTS

We have extended the charge quantization prescription suggested in [6], by setting a geometric invariant given by the square of a 5D geometric product defined in (22). We have obtained that, under an extension of (23), given by the quantization condition (27), provides a relationship between the gravito-electric, the gravito-magnetic and the gravitational mass densities, which can take discrete values according to (28), (29) and (30). They lead us to a set of conditions compatible with an effective 4D DRNBH. We must remark that in our previous work[6] we check that the obtained gravito-magnetic currents are in presence of a stable gravito-magnetic monopole which is inside the RNBH, but the relativistic observer which is placed outside the BH is not sensitive to the gravito-magnetic charge of the BH, because they (the gravito-electric and gravito-magnetic charges) are causally disconnected. For this reason in that case the gravito-electric and gravito-magnetic currents are orthogonal. However, in this work we have studied a different reality by dealing with a DRNBH in which both, Q_e and Q_m , coexist in a same causally connected region of space-time and for this reason $^{(ge)}\underline{J}$ and $^{(gm)}\overrightarrow{J}$ cannot be considered as orthogonal, as is the case of a RNBH. Of course, the study of duality between the interior and exterior space-time, and their physical properties, deserves a more profound study.

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⁶ The Latin index $n, m, p = 1...5$ are associated to the 5D manifold taking values along the five coordinates, the Greek index $\alpha, \beta = 1...4$ are associated to the effective 4D space-time taking values along the four effective coordinates obtained after the foliation.

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